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John Bongaarts
*Population Council*

Griffith Feeney



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John Bongaarts
Griffith Feeney

John Bongaarts is Vice President, Policy Research Division, Population Council, New York. Griffith Feeney is a demographer and consultant based in Scarsdale, New York.

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**Abstract**

The life expectancy implied by current age-specific mortality rates is calculated with life table methods that are among the oldest and most fundamental tools of demography. We demonstrate that these conventional estimates of period life expectancy are affected by an undesirable "tempo effect." The tempo effect is positive when the mean age at death is rising and negative when the mean age is declining. Estimates of the effect for females in three countries with high and rising life expectancy range from 1.6 years in the United States and Sweden to 2.4 years in France for the period 1980–95.

When a group of persons is observed from birth to death, mean lifetime may be calculated simply and directly as mean age at death. This statistic is problematic, however, for studying trends in mean lifetime. Mean lifetime for Swedish females born in 1850, for example, reflects mortality conditions from the mid-19th to the mid-20th century, a period of historically unprecedented increases in human survival. To study these changes requires a different approach.

Period life expectancy at birth calculated by life table methods has been the standard solution to this problem since the mid-19th century (Preston et al., 2001). This paper argues that it is an imperfect solution because life expectancy at birth calculated in this way is distorted whenever it is changing.

Conventional life expectancy depends solely on the force of mortality function for time $t$. We propose an alternative measure that depends both on the force of mortality function and on the rate of change in the standardized mean age at death. Our alternative is based on the assumption that the observed force of mortality function at any given time has the same shape as the force of mortality function inherent in the standardized population age distribution at time $t$, which reflects the history of mortality in the population. We demonstrate that this assumption is realistic in contemporary societies with high life expectancy, and also that the proposed measure is consistent with well-established measures used in other demographic contexts.

## Cohort Mean Lifetime

The distribution of lifetimes for a group of persons born during any given time period (a "birth cohort") may be described in three different ways. The *survival function*

$$\ell(a),\ a \geq 0 \tag{1a}$$

gives the proportion of individuals who survive to exact age $a$. It is non-increasing, with $\ell(0) = 1.0$ and $\ell(\omega) = 0$ for some advanced age $\omega$. The *death density* function

$$d(a) \equiv \frac{-\partial \ell(a)}{\partial a} \tag{1b}$$

gives the distribution of deaths by age. The *force of mortality* function

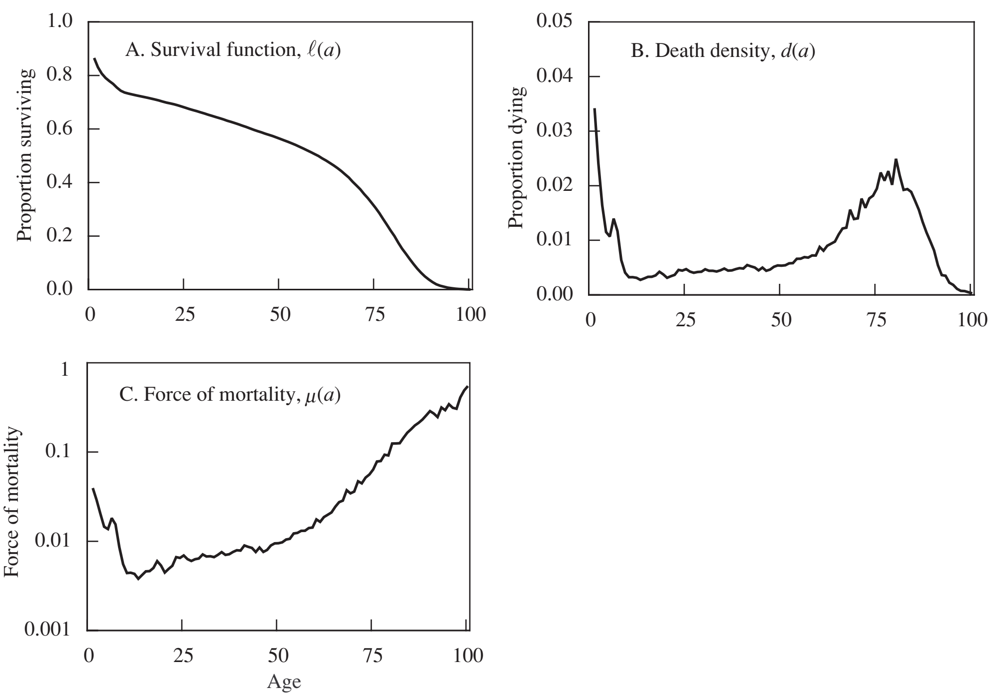
$$\mu(a) \equiv \frac{d(a)}{\ell(a)} = \frac{-\partial \ell(a) / \partial a}{\ell(a)} \tag{1c}$$

gives the risk of dying at each age. These functions are formally equivalent in the sense that any two may be derived from the third. The force of mortality function $\mu(a)$ may be derived from $d(a)$ or $\ell(a)$ using (1c), for example, and $\ell(a)$ may be derived from $\mu(a)$ or $d(a)$ using

$$\ell(a) = \int_a^{\omega} d(x)dx = \exp[-\int_0^a \mu(x)dx]. \tag{1d}$$

Figure 1 plots $\ell(a)$, $d(a)$, and $\mu(a)$ for the cohort of females born in Sweden in 1850. The survival function declines to zero at around age 100 years. The density function is broadly bi-modal with peaks at age 0 and approximately 80 years. The force of

**Figure 1** Mortality experience of the cohort of Swedish females born in 1850, as summarized by the survival function, $\ell(a)$ (A), the death density function $d(a)$ (B), and the force of mortality function $\mu(a)$ (C)



mortality exhibits a U-shaped pattern with a minimum at about age 10. Note the use of the log scale to accommodate the large differences in magnitude at different ages. These patterns are broadly typical, though levels of mortality vary widely between populations and over time.

Mean lifetime for a birth cohort, $M$, may be calculated from $\ell(a)$ as

$$\int_0^\infty \ell(a)da, \tag{2a}$$

from $d(a)$ as

$$\int_0^\infty ad(a)da, \tag{2b}$$

or from $\mu(a)$ as

$$\int_0^\infty \left\{\exp[-\int_0^a \mu(x)dx]\right\}da. \tag{2c}$$

These formulas give identical results. For the 1850 cohort of Swedish females, for example, we calculate $M = 48.1$ years from each.

## PERIOD MEAN LIFETIME

Let

$$\ell(a,t) \equiv \ell_{t-a}(a), \tag{3a}$$

$$d(a,t) \equiv d_{t-a}(a), \text{ and} \tag{3b}$$

$$\mu(a,t) \equiv \mu_{t-a}(a), \tag{3c}$$

where the subscripts at right indicate time of birth. Thus $\ell(a,t)$ denotes the proportion of persons born at time $t-a$ who are surviving at time $t$, $d(a,t)$ denotes the density of deaths for this cohort at age $a$ and time $t$, and $\mu(a,t)$ denotes the corresponding force of mortality. Note that $\ell(a,t)$ and $d(a,t)$ differ from the survival and density functions for synthetic cohorts obtained from conventional period life tables, and that their calculation requires data either on past births and migrations or on past deaths.

We refer to $\ell(a,t)$ as the standardized population age distribution at time $t$ and to $d(a,t)$ as the standardized age distribution of deaths at time $t$. The standardized population age distribution and age distribution of deaths are the same as their unstandardized counterparts in any population that experiences constant numbers of births over time.

By analogy with (2), mean lifetime at time $t$ may be calculated as

$$M_1(t) = \int_0^\infty \ell(a,t)da \text{, as} \tag{4a}$$

$$M_2(t) = \frac{\int_0^\infty a d(a,t)da}{\int_0^\infty d(a,t)da} \text{, or as} \tag{4b}$$

$$M_3(t) = \int_0^\infty \exp[-\int_0^a \mu(x,t)dx]\,da\,. \tag{4c}$$

Each of these formulas has been used in demography to calculate period mean age for some demographic event. Mean age at first marriage is often calculated as a variant of $M_1(t)$ that allows for persons not marrying. This is the singulate mean age at marriage introduced by Hajnal (1953), with $\ell(a,t)$ taken as the proportion of single persons at age $a$ at time $t$ (see for example United Nations, 1990). Mean age at childbearing is generally calculated as $M_2(t)$, with age-specific or age-order-specific birth rates substituted for $d(a,t)$ (see for example Council of Europe, 2001). Life expectancy at birth, denoted $e_0(t)$, is conventionally calculated as $M_3(t)$.

We refer to $M_2(t)$ as the standardized mean age at death. The unstandardized mean age at death is unacceptable as a measure of mean lifetime because it may be heavily distorted by the population age distribution. This objection does not apply to the standardized mean age at death, which might be a widely used measure of period mean lifetime if it were more easily calculated.

If $\ell(a,t)$ is constant with respect to $t$, the three means defined by (4) are identical. When length of life changes, the three means diverge. The following sections develop relationships between them.

## Relationship Between $M_1$ and $M_2$

To establish a simple relationship between $M_1(t)$ and $M_2(t)$ let

$$d_s(a,t) = \frac{-\partial \ell(a,t)}{\partial a} \text{ and } \mu_s(a,t) = \frac{d_s(a,t)}{\ell(a,t)}. \tag{5a-b}$$

The age schedules $d_s(a,t)$ and $\mu_s(a,t)$ are inherent in the standardized population age distribution at time $t$. They may be interpreted as the age distribution of deaths and the force of mortality function in the stationary population whose age distribution is given by $\ell(a,t)$, with $\ell(0,t) = 1$ for all $t$. This interpretation is of course valid only if the mortality history of the population is such that $\ell(a,t)$ is a non-increasing function of $a$ ($d\ell(a,t)/da \leq 0$).

Assume now that for $t$ in the time interval $[0,\Delta]$ there exists a function $p(t)$ independent of age such that

$$\mu(a,t) = p(t)\mu_s(a,t) \tag{6a}$$

or, equivalently,

$$d(a,t) = p(t)d_s(a,t), \tag{6b}$$

and that the function $p(t)$ is a real valued integrable function bounded below by 0. We refer to this as the *proportionality assumption*.

The proportionality assumption implies that the age schedules of $\mu(a,t)$ and $d(a,t)$ are the same in shape (but not necessarily in level) as the age schedules of $\mu_s(a,t)$ and $d_s(a,t)$. As will be shown below, this assumption provides a good approximation for patterns of adult mortality in contemporary countries with high life expectancy.

From (4a) and (5a),

$$M_1(t) = \int_0^\infty \ell(a,t)da = \frac{\int_0^\infty a d_s(a,t)da}{\int_0^\infty d_s(a,t)da}, \tag{7a}$$

and from (4b) and (6b),

$$M_2(t) = \frac{\int_0^\infty a p(t) d_s(a,t) da}{\int_0^\infty p(t) d_s(a,t) da}. \tag{7b}$$

On cancellation of the proportionality factor $p(t)$, (7b) becomes (7a), thus proving that $M_1(t) = M_2(t)$.

## OTHER IMPLICATIONS OF THE PROPORTIONALITY ASSUMPTION

It is shown in Appendix A that if the proportionality assumption holds, then

$$p(t) = 1 - \frac{\partial M_1(t)}{\partial t}. \tag{8a}$$

Substituting this in (6) and noting that $M_1(t) = M_2(t)$ yields

$$\mu(a,t) = \left[1 - \frac{\partial M_2(t)}{\partial t}\right]\mu_s(a,t) \tag{8b}$$

$$d(a,t) = \left[1 - \frac{\partial M_2(t)}{\partial t}\right]d_s(a,t). \tag{8c}$$

This shows that $\mu(a,t)$ and $d(a,t)$ are functions of the rate of change in the standardized mean age at death $M_2(t)$, because $\mu_s(a,t)$ and $d_s(a,t)$ are determined by mortality conditions up to time $t$. When this mean age is rising, $\mu(a,t) < \mu_s(a,t)$ and $d(a,t) < d_s(a,t)$, but when it is declining $\mu(a,t) > \mu_s(a,t)$ and $d(a,t) > d_s(a,t)$.

As shown in Appendix B, the proportionality assumption also implies that the age schedule $\ell(a,t)$ shifts uniformly to older (younger) ages as the mean age at death rises (falls). Uniform shifting between time 0 and time $T$ means that there is a function $F(t) = M_1(t) - M_1(0)$, giving the magnitude of the shift between time 0 and time $t$, such that, for all $0 \le t \le T$,

$$\ell(a,t) = \ell(a - F(t), 0) \text{ for all } a \ge F(t), \tag{9}$$

and $\ell(a,t) = 1$ for $a < F(t)$. Downward as well as upward shifts are possible provided that $\ell(a,t) = 1$ for $a$ less than some number greater than 0.

It follows from (5) that uniform shifts in $\ell(a,t)$ imply uniform shifts in $\mu_s(a,t)$ and $d_s(a,t)$ with the same shift function $F(t)$, with $\mu_s(a,t) = d_s(a,t) = 0$ when $\ell(a,t) = 1$. The proportionality assumption is therefore equivalent to the shifting assumption made by Bongaarts and Feeney (2002).

Changes over time in the schedules $\mu(a,t)$ and $d(a,t)$ are of two types. First, as the mean age at death rises or falls, $\mu(a,t)$ and $d(a,t)$ shift to higher or lower ages with $\ell(a,t)$, $\mu_s(a,t)$, and $d_s(a,t)$. Second, $\mu(a,t)$ and $d(a,t)$ are deflated or inflated relative to $\mu_s(a,t)$ and $d_s(a,t)$ by the proportionality factor $p(t)$.

## MORTALITY CHANGE IN FRANCE, SWEDEN, AND THE UNITED STATES

We will now show that observed mortality patterns conform closely to the proportionality assumption (6) if we ignore child and young adult mortality. All quantities in this section and in Figures 2 through 6 and Table 1 are calculated from observed values of $\mu(a,t)$ for ages above 30, but $\mu(a,t)$ is set to zero for ages under 30 years for all $t$. Our estimates of life expectancy at birth are therefore equal to 30 plus the life expectancy at age 30. For populations with high life expectancy, nearly all deaths (97–98 percent) occur at ages over 30 years, and actual life expectancy at birth is therefore close to 30 years plus the life expectancy at age 30.

The plots at left in Figure 2 show the age schedules $\mu(a,t)$, $\mu_s(a,t)$, and $p(t)\mu_s(a,t)$, all calculated as averages of annual values for 1980–95, for France, Sweden, and the United States. The plots at right show the age schedules $d(a,t)$, $d_s(a,t)$, and $p(t)d_s(a,t)$, calculated in the same way, with $p(t)$ estimated with equation (8a). The near coincidence of $\mu(a,t)$ and $p(t)\mu_s(a,t)$, and of $d(a,t)$ and $p(t)d_s(a,t)$, shows that the proportionality assumption is a good approximation for all three countries. Note that the logarithmic scale used at left means that perfect proportionality corresponds to constant differences between the plotted values of $\mu(a,t)$ and $\mu_s(a,t)$.

The plots at left in Figure 3 show $\mu(a,t)$ for 1980 and 1995 for the same three countries. The plots at right show corresponding values for $d(a,t)$. The pattern of change in these schedules is consistent with the pattern of shifting and inflation/deflation noted above.

**Figure 2** Average force of mortality for 1980–95, observed as $\mu(a,t)$, estimated from $\ell(a,t)$ as $\mu_S(a,t)$, and estimated as the product $\mu_S(a,1980\text{–}95)p(t)$ for France (A), Sweden (B), and the United States (C). Average death density function for 1980–95, observed as $d(a,t)$, estimated from $\ell(a,t)$ as $d_S(a,t)$, and estimated as the product $d_S(a,1980\text{–}95)p(t)$ for France (D), Sweden (E), and the United States (F)

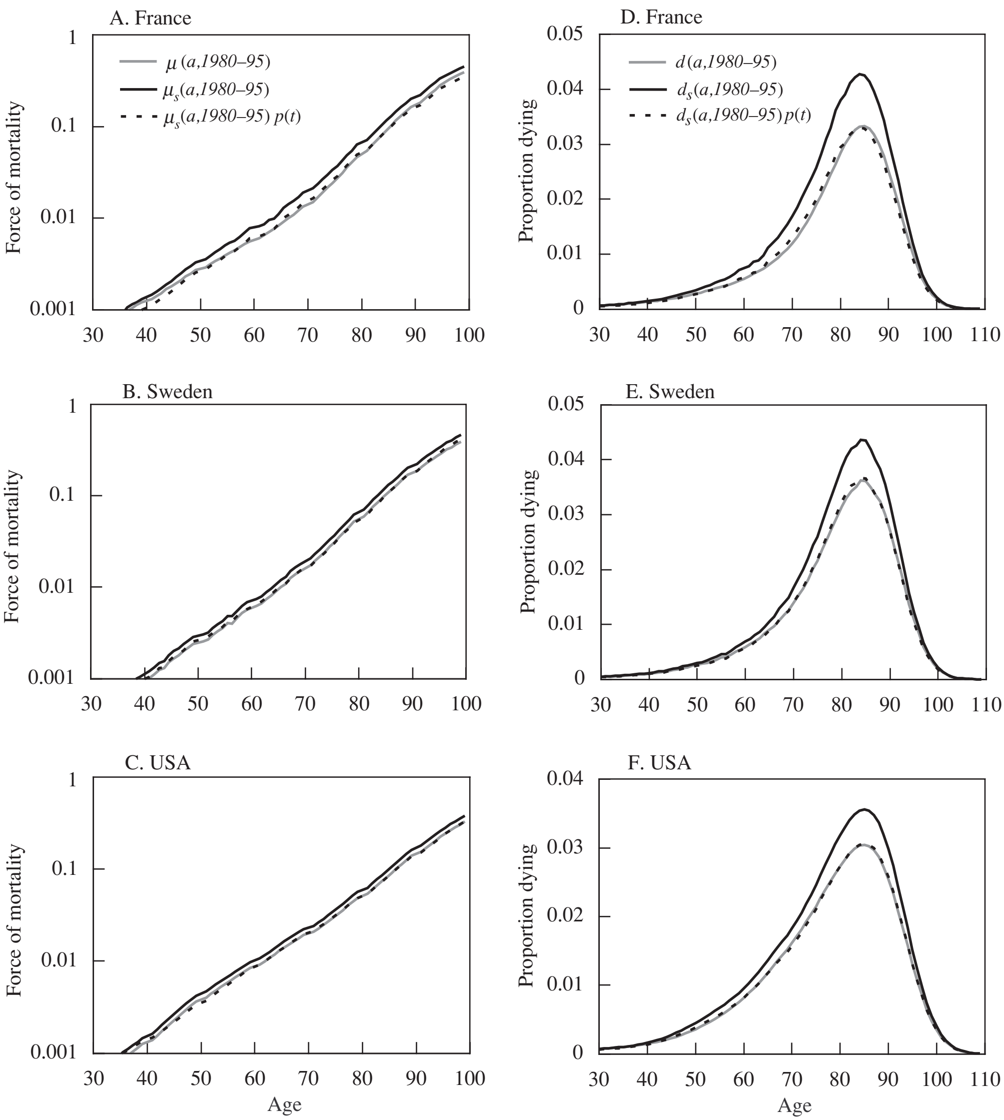


**Figure 3** Observed period force of mortality $\mu(a,t)$ in 1980 and 1995 for France (A), Sweden (B), and the United States (C). Observed period death density function $d(a,t)$ in 1980 and 1995 for France (D), Sweden (E), and the United States (F)

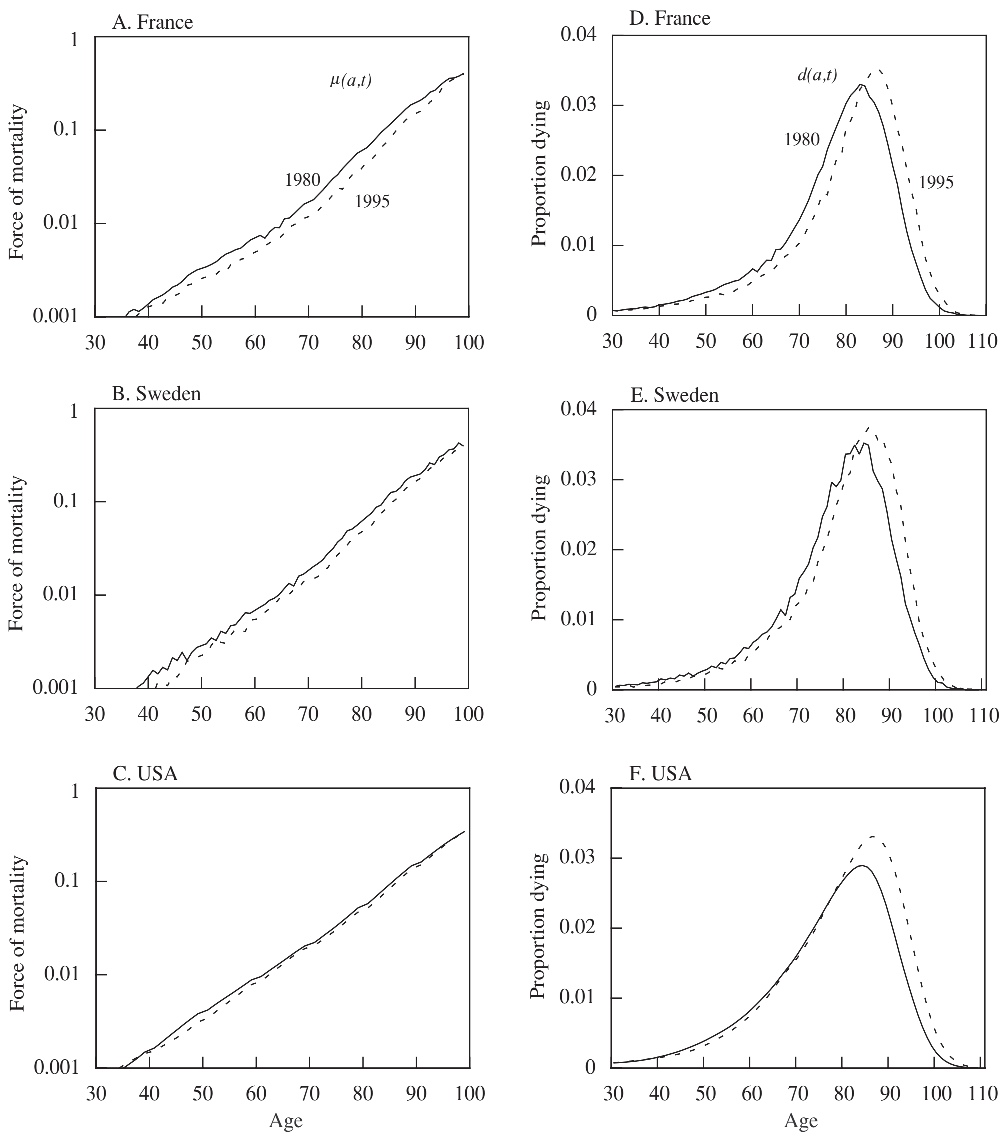


**Figure 4** Observed period survival function $\ell(a,t)$ in 1980 and 1995 for France (A), Sweden (B), and the United States (C)

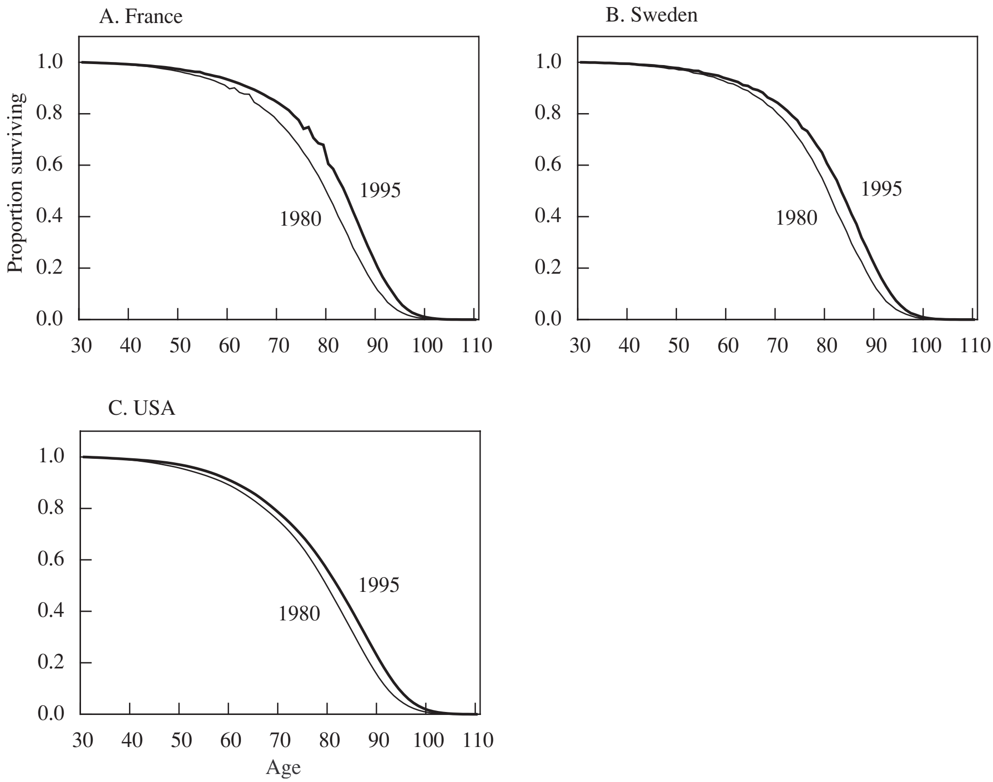



Figure 4 plots the age schedule $\ell(a,t)$ for 1980 and 1995 for the three countries. The shape of $\ell(a,t)$ changes very little, but there is a shift to higher ages as life expectancy rises. The magnitude of the shift was 3.4 years for France, 2.4 years for Sweden, and 2.1 years for the United States.

The first three columns of Table 1 present averages of annual estimates of $M_1(t)$, $M_2(t)$, and $M_3(t)$ for the years 1980–95. The values for $M_1(t)$ and $M_2(t)$ are nearly identical, as expected, but the $M_3(t)$ values are substantially higher. The reason for the higher value of $M_3(t)$ is discussed in following sections.

**Table 1** Alternative estimates of the period mean age at death, assuming no mortality under age 30

| | Mean age at death, females, 1980–95 | | | | |
|---|---|---|---|---|---|
| | $M_1(t)$ | $M_2(t)$ | $M_3(t)$ $(= e_0(t))$ | $M_4(t)$ | *Tempo effect* $M_3(t) - M_4(t)$ |
| France | 79.0 | 79.2 | 81.4 | 79.0 | 2.4 |
| Sweden | 79.5 | 79.5 | 81.1 | 79.4 | 1.6 |
| USA | 78.3 | 78.3 | 79.9 | 78.3 | 1.6 |

**Source:** Death rates from University of California, Berkeley Mortality Data Base

## Tempo Effects in Demographic Analysis

Tempo effects were first discovered and analyzed in the study of fertility. If women shift the ages at which they bear children upward without changing their completed fertility, annual numbers of births will be less than they would otherwise have been because the same number of births will be spread out over a longer time period. Similarly, if women begin to have children at younger ages, annual numbers of births will be larger than they would otherwise have been because the same number of births occur over a shorter time period. These changes in annual number of births induced by changes in the timing of childbearing are *tempo effects*.

Fertility tempo effects have been extensively documented. The postwar "baby boom" in the United States, for example, was due in part to a decline in the mean age at childbearing during the late 1940s and the 1950s (Hajnal, 1947; Ryder, 1964, 1980; Bongaarts and Feeney, 1998).

Tempo effects complicate the study of levels and trends of fertility because they produce changes in period fertility rates that depend on the rate at which the mean age at childbearing changes, independently of changes in completed fertility of cohorts. Ryder (1956) introduced the term "timing distortion" to refer to tempo effects because they are undesirable in most analyses of fertility levels and trends.

Tempo effects influence demographic processes other than fertility. A tempo effect can be defined in general as an inflation or deflation of the period incidence of a demographic event (births, marriages, deaths) resulting from a rise or fall in the mean age at which the event occurs.

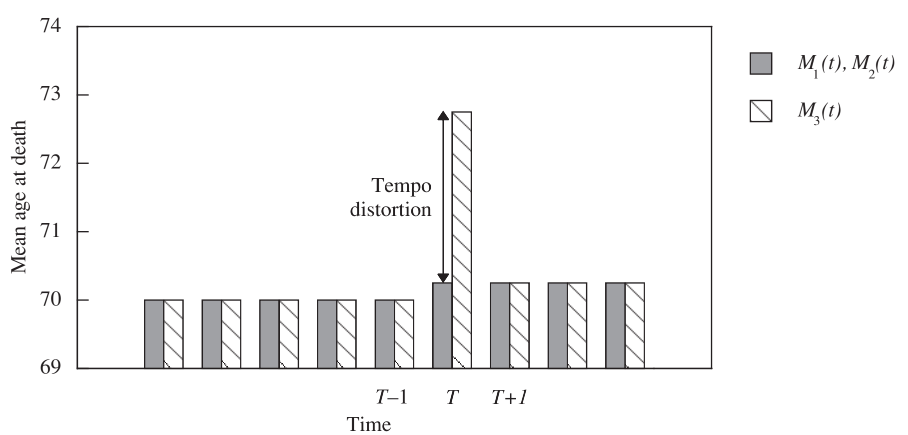
## TEMPO EFFECTS IN MORTALITY

A simple example will demonstrate how mortality tempo effects operate. Consider a stationary population with a life expectancy at birth of 70 years. Suppose that the exact age of death of each individual is predetermined until the invention of a "life extension" pill that adds 3 months to the life of any person who consumes it.

If everyone in the population takes this pill on January 1 of year *T*, there will be no deaths during the first three months of the year. The number of deaths in year *T* will fall by 25 percent and the mean age at death will rise from 70 to 70.25 years. Since the pill's effect is the same at all ages, the level of the force of mortality function is also reduced by 25 percent, and the age to which each value of the function is attached increases by 0.25 years. This fall in values of the force of mortality function, together with the shift to older ages, causes life expectancy at birth as conventionally calculated to rise to nearly 73 years for year *T*.

In the following year, the number of deaths and the force of mortality function rise to the level observed before year *T*, but with values shifted forward to older ages by 0.25 years. Life expectancy at birth as conventionally calculated, having risen from 70

**Figure 5** Hypothetical illustration of the effect of an increase in mean age at death by 0.25 years (from 70.0 to 70.25) during year *T* on conventional life expectancy. Before and after *T*, $M_1(t)$=$M_2(t)$=$M_3(t)$. During *T*, a tempo distortion of –25% in the number of deaths results in an upward distortion of approximately 2.5 years in $M_3(t)$



years prior to year $T$ to nearly 73 years during year $T$, falls back to 70.25 years (Figure 5). We contend that this rise and fall in life expectancy at birth as conventionally calculated is a tempo distortion because it is at variance with the known trend in the mean length of life. Distortion of this kind occurs whenever the standardized mean age at death changes.

## Removing Tempo Effects

The tempo effect deflates (inflates) $d(a,t)$ and $\mu(a,t)$ when the standardized mean age at death rises (falls). Formulas (8b-c) show that this deflation or inflation is estimated by the multiplicative factor $1-\partial M_2(t)/\partial t$ when the proportionality assumption holds. The tempo effect may therefore be removed by dividing $d(a,t)$ and $\mu(a,t)$ by $1-\partial M_2(t)/\partial t$. Since $M_1(t) = M_2(t)$, division by $1-\partial M_1(t)/\partial t$ gives the same result. The latter approach is preferred because it gives more stable results when applied to observed mortality rates. We define

$$\mu^*(a,t) = \mu(a,t)/(1-\partial M_1(t)/\partial t) \text{ and} \tag{10a}$$

$$d^*(a,t) = d(a,t)/(1-\partial M_1(t)/\partial t) \tag{10b}$$

and refer to the expressions on the left as the *tempo-adjusted* death density and force of mortality. It follows from (8) that $\mu^*(a,t) = \mu_s(a,t)$ and $d^*(a,t) = d_s(a,t)$ when the proportionality assumption holds.

To calculate life expectancy at birth corrected for the tempo effect, we use the defining formula (4c) with $\mu^*(a,t)$ substituted for $\mu(a,t)$, giving

$$\begin{aligned} M_4(t) &= \int_0^\infty \exp\left\{-\int_0^a [\mu(x,t)/(1-\partial M_1(t)/\partial t]dx\right\}da \\ &= \int_0^\infty \exp\left\{-\int_0^a \mu_s(a,t)dx\right\}da \\ &= \int_0^\infty \ell(a,t)da \\ &= M_1(t), \end{aligned} \tag{11}$$

where $M_4(t)$ denotes life expectancy at birth without the tempo effect. Removing the tempo effect from $M_3(t)$ gives the same result as $M_1(t)$ or $M_2(t)$. The undistorted life expectancy at birth can be estimated as $M_1(t)$, $M_2(t)$, or $M_4(t)$.

**Figure 6** Trends from 1980 to 1995 for alternative estimates of mean age at death $M_1$, $M_2$, $M_3$, and $M_4$ for France (A), Sweden (B), and the United States (C). The difference $M_3$–$M_4$ equals the tempo effect.

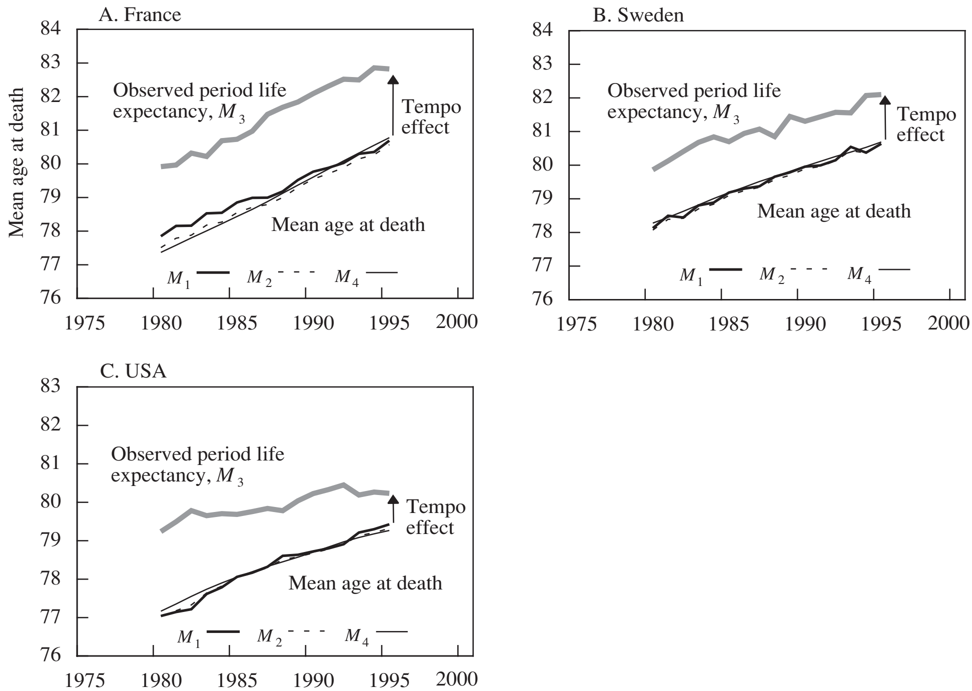



Table 1 shows average annual values of $M_4(t)$ as well as $M_1(t)$, $M_2(t)$, and $M_3(t)$ for females in France, Sweden, and the United States for the period 1980–95. The corresponding annual trends are plotted in Figure 6. These results confirm that $M_1(t)$, $M_2(t)$, and $M_4(t)$ are nearly identical, but $M_3(t)$, the life expectancy at birth calculated by conventional life table methods, is substantially higher than the other three means. The tempo effect, $M_3(t)$ minus $M_4(t)$, averages 2.4 years for France and 1.6 years for Sweden and the United States.

This analysis of tempo effects is based on trends in adult mortality only. We ignore any tempo effects in mortality under age 30, because they are probably small and difficult to quantify. In the absence of tempo effects under age 30, the tempo effect in life expectancy at birth is only 2 or 3 percent smaller than the tempo effect above age 30 measured here. This is because the probability of survival from birth to age 30 is typically 0.97–0.98 in contemporary societies with high life expectancy.

## CONCLUSION

Life expectancy at birth as conventionally calculated is distorted whenever it is changing. We have provided formulas to adjust for this distortion. The formulas are applicable to populations with high life expectancy. The adjustments for France, Sweden, and the United States in recent decades reduce conventionally calculated life expectancy at birth by 1.6 to 2.4 years. These results confirm and extend those given in Bongaarts and Feeney (2002).

The essential argument is as follows. Empirical observation indicates that the proportionality assumption is closely approximated when life expectancy at birth is high and child and young adult mortality are ignored. When the proportionality assumption holds, increases (decreases) in length of life are realized by a uniform translation of the standardized population age distribution and the force of mortality function inherent in this age distribution to higher (lower) ages. Neither the shape nor the level of the standardized age distribution or the inherent force of mortality function changes, only their location on the age scale.

The force of mortality function is likewise translated to higher (or lower) ages without any change in shape, but its level changes with the rate of change in the standardized mean age at death as shown by (8b). When the standardized mean age at death rises (falls), the force of mortality function falls and shifts to the right (rises and shifts to the left). This fall (rise) in the force of mortality represents the tempo effect, and it produces an undesirable rise (fall) in life expectancy at birth as conventionally calculated. In our hypothetical example (Figure 5), increasing the standardized mean age at death from 70 to 70.25 years over one year results in a temporary decline of 25 percent in the force of mortality function and a temporary rise of nearly 3 years in conventionally calculated life expectancy at birth. The tempo effect in life expectancy in this case is about 10 times the net change in mean lifetime.

In interpreting these findings it is important to distinguish between current observed death rates and current mortality conditions (Vaupel, 2002). We do not question the conventional life table calculation of period life expectancy from observed age-specific death rates. We argue rather that tempo effects distort both the observed death rates and the corresponding life expectancy, so that their values give a misleading indication of current mortality conditions.

Our empirical focus has been on human survival, but life table methods are widely applied to survival data of all kinds. Examples include age at marriage (the interval between birth and marriage), birth interval analysis (intervals between successive births), length of schooling (interval between entering and leaving school), and postoperative survival (interval between operation and death). It is therefore likely that tempo effects are pertinent to many other kinds of statistical survival analyses.

## APPENDIX A

We have to prove that the proportionality assumption (6) implies formula (8a) of the text. Bennett and Horiuchi (1981), Preston and Coale (1982), and Arthur and Vaupel (1984) show that

$$\mu(a,t) = \mu_s(a,t) - r(a,t), \tag{A1}$$

where

$$r(a,t) = \frac{-\partial\ell(a,t)/\partial t}{\ell(a,t)} \tag{A2}$$

is the age-specific growth rate for age $a$ at time $t$ for the population whose age distribution at time $t$ is given by $\ell(a,t)$. Note that (A1) may be written as

$$\mu(a,t) = -\left[\frac{\partial\ell(a,t)/\partial a}{\ell(a,t)} + \frac{\partial\ell(a,t)/\partial t}{\ell(a,t)}\right], \tag{A3}$$

which is an equation used in modeling cell population dynamics (McKendrick, 1926; Von Foerster, 1959; Trucco, 1965 a, b).

Equating the expressions for $\mu(a,t)$ given by the proportionality assumption (6a) and (A1) and rearranging terms gives

$$r(a,t) = [1-p(t)]\mu_s(a,t). \tag{A4}$$

Substitution of (A2) and text formula (5b) in (A4) yields

$$\frac{\partial \ell(a,t)}{\partial t} = [1 - p(t)]\frac{-\partial \ell(a,t)}{\partial a}. \tag{A5}$$

From the definition (4a) of $M_1(t)$, then,

$$\frac{\partial M_1(t)}{\partial t} = \frac{\partial}{\partial t}\int_0^\infty \ell(a,t)da = \int_0^\infty \frac{\partial \ell(a,t)}{\partial t}da = [1 - p(t)]\int_0^\infty \frac{-\partial \ell(a,t)}{\partial a}da. \tag{A6}$$

Since the last integral on the right equals one, we have established formula (8a) of the text.

Integrating the density function $d(a,t)$ over age results in a period mortality measure that may be called the total mortality rate *TMR(t)*. (This measure is equivalent to the total fertility rate, which is widely used in the analysis of fertility levels and trends.)

$$TMR(t) = \int_0^\infty d(a,t)da. \tag{A7}$$

Substitution of (8a) gives

$$TMR(t) = \int_0^\infty p(t)d_s(a,t)da = p(t). \tag{A8}$$

## APPENDIX B

We have to prove that the proportionality assumption implies uniformly shifting age distributions, i.e., formula (9) of the text, provided there is no mortality at younger ages. The first step is to find a characterization of uniformly shifting age distributions that applies to a point in time. The directional derivative provides such a characterization. The directional derivative of the function $\ell(a,t)$ at the point $(a,t)$ in the direction $(b,u)$ is the rate of change at time $t$ of the function $\ell(a+bt,t+ut)$, which may be expressed as

$$\frac{1}{\sqrt{b^2+u^2}}\left[b\frac{\partial \ell(a,t)}{\partial a} + u\frac{\partial \ell(a,t)}{\partial t}\right]. \tag{A9}$$

Now let $f(a,t)$ be such that the directional derivative of $\ell(a,t)$ at the point $(a,t)$ in the direction $(f(a,t),1)$ equals zero. Uniform translation corresponds to the condition that $f(a,t)$ be constant with respect to age, $f(a,t) \equiv f(t)$ for all $t$, and therefore to the condition

$$f(t)\frac{\partial \ell(a,t)}{\partial a} + \frac{\partial \ell(a,t)}{\partial t} = 0. \qquad \text{(A10)}$$

If this identity holds, the directional derivative of $\ell(a,t)$ at the point $(a,t)$ in the direction $(f(t),1)$ is zero.

If the proportionality assumption holds, text formula (8b) holds (as just shown in Appendix A), and this together with (A1) implies, equating the expressions for $\mu(a,t)$ and rearranging terms,

$$\frac{\partial M_1(t)}{\partial t}\mu_s(a,t) - r(a,t) = 0. \qquad \text{(A11)}$$

Multiplying both sides by $-\ell(a,t)$ gives

$$\frac{\partial M_1(t)}{\partial t}\frac{\partial \ell(a,t)}{\partial a} + \frac{\partial \ell(a,t)}{\partial t} = 0, \qquad \text{(A12)}$$

which shows that the directional derivative of $\ell(a,t)$ at $(a,t)$ in the direction $(f(t),t)$ equals 0 for all ages $a$, with $f(t) = \partial M_1(t)/\partial t$.

To show that this implies uniform shifting of the age distribution, it is necessary only to note that $f(t)$ is the rate of change of the contour line in the age-time plane defined by the points $(x+t,t)$ for which $\ell(x+t,t) = \ell(a,0)$. The function $F(t)$ of the uniform shifting formula (9) of the text therefore equals the integral of $f(\cdot)$ from 0 to $t$.

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